Solution of the Hurwitz problem with a length-2 partition

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Abstract

In this note we provide a new partial solution to the Hurwitz existence problem for surface branched covers. Namely, we consider candidate branch data with base surface the sphere and one partition of the degree having length two, and we fully determine which of them are realizable and which are exceptional. The case where the covering surface is also the sphere was solved somewhat recently by Pakovich, and we deal here with the case of positive genus. We show that the only other exceptional candidate data, besides those of Pakovich (five infinite families and one sporadic case), are a well-known very specific infinite family in degree 4 (indexed by the genus of the candidate covering surface, which can attain any value), five sporadic cases (four in genus 1 and one in genus 2), and another infinite family in genus 1 also already known. Since the degree is a composite number for all these exceptional data, our findings provide more evidence for the prime-degree conjecture. Our argument proceeds by induction on the genus and on the number of branching points, so our results logically depend on those of Pakovich, and we do not employ the technology of constellations on which his proof is based.

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A *surface branched cover* is a map $f:\Theta\to\Sigma$, where Θ and Σ are connected closed and orientable surfaces and f is locally modeled on the function $\mathbb{C}\ni z\mapsto z^k\in\mathbb{C}$ for a positive integer k. If $k>1$ the point corresponding to 0 in the target $\mathbb{C}$ is called a *branching point*, and k is called the *local degree* of f at the point corresponding to 0 in the source $\mathbb{C}$. There is a finite number n of branching points, and removing all of them and their preimages one gets an ordinary cover of some degree d, called the *degree* of f. The collection of the local degrees at the preimages of the j-th branching point forms a partition π_j of d, namely an unordered array of positive integers summing up to d, possibly with repetitions. To such an f we associate a symbol called a *branch datum* given by $(\Theta,\Sigma,d,n;\pi_1,\ldots,\pi_n)$ where

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the partitions $\pi_1, \ldots, \pi_n$ are viewed up to reordering. If ℓ_j denotes the length of π_j, the datum satisfies the Riemann-Hurwitz condition

$$\chi(\Theta) - (\ell_1 + \ldots + \ell_n) = d \cdot (\chi(\Sigma) - n). \tag{1}$$

We now call *candidate branch datum* a symbol $(\Theta, \Sigma, d, n; \pi_1, \ldots, \pi_n)$ with Θ and Σ connected closed orientable surfaces, d and n positive integers, and $\pi_1, \ldots, \pi_n$ partitions of d, satisfying condition (1). We will always assume that no π_j is the trivial partition with all entries 1. A candidate branch datum is *realizable* if it is the branch datum associated to an existing surface branched cover f, and *exceptional* otherwise.

The question of characterizing which candidate branch data are realizable and which are exceptional is known as the *Hurwitz existence problem* [10]. It has a long history (see the surveys [25, 29] and [2, 3, 4, 6, 7, 8, 11, 12, 20, 21, 22, 23, 24, 31, 32]), and many motivations (see for instance [19]). Before proceeding we fix the following:

Notation We indicate by S the sphere, by T the torus and by $g \cdot T$ the orientable connected closed surface of genus g for $g \geqslant 2$, but we also accept the symbols $0 \cdot T = S$ and $1 \cdot T = T$. We use double square brackets $[\![\cdot]\!]$ to denote an unordered array of objects with possible repetitions. So a partition π of a positive integer d is given by $\pi = [\![q_1, \ldots, q_m]\!]$ where the q_i's are positive integers and $q_1 + \ldots + q_m = d$. We denote by $\ell(\pi)$ the length m of π.

Some results from the literature We cite a crucial known [5] result:

Theorem 0.1. *Every candidate branch datum $(\Theta, g \cdot T, d, n; \pi_1, \ldots, \pi_n)$ with $g \geqslant 1$ is realizable.*

This implies that to find a full solution of the Hurwitz existence problem one is only left to consider the case where the candidate covered surface is the sphere S (see also Remark 0.4 below). Many different techniques were employed over time to attack the problem in this case, and a huge variety of exceptional and realizable candidate branch data were detected (see the reference already cited above). We refrain from giving a full account of these results here, confining ourselves to those that are most relevant for the present paper. We start with the following (see [32] for the case $g = 0$ and [5, Proposition 5.2] for any g):

Theorem 0.2. *A candidate branch datum $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ such $\ell(\pi_j) = 1$ for some j, namely $\pi_j = [\![d]\!]$, is always realizable.*

One can informally phrase this result saying that a candidate branch datum with one partition *as short as it could at most be* is realizable. It is then natural to consider the case where one partition has the next shortest possible length, *i.e.* 2. Namely one can ask the question of which candidate branch data $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ such that $\ell(\pi_j) = 2$ for some j, that is $\pi_j = [\![s, d - s]\!]$ for $0 < s < d$, are realizable. The solution was obtained in the following cases:

- For any g, any n and $s = 1$ in [5];
- For $n = 3$, any s and $\pi_2 = \pi_3 = [\![2, \ldots, 2]\!]$, whence $g = 0$, in [5];
- For any g, $n = 3$ and $s = 2$ in [24];
- For $g = 0$, any n and any s in [20].

The aim of the present paper is to provide a complete answer to the question, namely to face the case of any $g \geqslant 1$, any n and any s left out by Pakovich [20]. The (long) answer will be given in Section 3 (see Theorem 3.1, whose statement does not require any of the notions treated in Sections 1 and 2). We only mention here that our argument is based on two inductions, one on g for fixed $n = 3$, with the base step $g = 0$ given by the statement of [20], and one on n. In particular, our proof depends on that of Pakovich for $n = 3$, and it does not employ the technology of constellations he uses.

The prime-degree conjecture The following was proposed in [5] and served ever since as a guiding idea in this area of research:

Conjecture 0.3. *A candidate branch datum $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ with prime d is always realizable.*

It was also shown in [5] that proving the conjecture with $n = 3$ would imply its validity for all n. All the results obtained so far concerning the Hurwitz existence problem turned out to be compatible with this conjecture (and some of them actually provided striking supporting evidence for it, see for instance [21]). Our Theorem 3.1 below makes no exception.

Concluding remarks We end this introduction with some additional considerations.

Among the various approaches to the Hurwitz existence problem, a remarkable one was proposed (for arbitrary n) by Zheng [33] in computational terms, that he implemented on a machine for $n = 3$, giving a complete classification of realizable and exceptional candidate branch data with $n = 3$ and $d \leqslant 20$. The first named author pushed the implementation of the methods of [33], still for $n = 3$, up to degree $d \leqslant 29$ (confirming the truth of the prime-degree conjecture up to this level). In Sections 4 and 5 we will sometimes refer to computer-aided findings based on [33] (mostly for $n = 3$, but not only) to make our arguments more concise, but the realizability of the relevant candidate branch data can always also be very easily established by theoretical methods.

Remark 0.4. The Hurwitz existence problem can be stated also for candidate branch data $\mathcal{D} = (\Theta, \Sigma, d, n; \pi_1, \ldots, \pi_n)$ with non-orientable Σ and possibly non-orientable Θ. In this case the Riemman-Hurwitz condition (1) must be complemented with two more (one obvious for orientable Θ and one less obvious for non-orientable Θ). However the following facts proved in [5], with $\mathbb{P}$ denoting the projective plane, show that a full solution of the problem follows once it is given for $\Sigma = S$:

- If $\Sigma = k \cdot \mathbb{P}$ with $k \geqslant 2$ then $\mathcal{D}$ is realizable;
- If $\Sigma = \mathbb{P}$ and Θ is non-orientable then $\mathcal{D}$ is realizable;
- If $\Sigma = \mathbb{P}$ and Θ is orientable the realizability of $\mathcal{D}$ is equivalent to that of one of a finite number of candidate branch data with candidate covered surface S.

Remark 0.5. The original question of Hurwitz was actually that of *how many* realizations of a given candidate datum exist, up to some natural equivalence relation. This problem was given a deep but somewhat implicit solution in [17, 18], from which it is not easy to extract a solution to the realizability problem for specific candidate branch data. See also [14, 15, 16], that also provide general but indirect answers, the more explicit [26, 27, 28] and the easy remarks contained in [30] on the different ways the realizations can be counted.

The paper is organized as follows. In Section 1 we introduce the idea of *reduction move*, which is crucial for our induction arguments, and we define some *topological* reduction moves, based on the notion of *dessin d'enfant.* In Section 2 we next introduce some *algebraic* reduction moves, using the monodromy approach to the Hurwitz existence problem. Then in Section 3 we state our result, listing all the exceptional candidate branch data with a length-2 partition, and explaining why they are indeed exceptional using the technology of Sections 1 and 2. As a conclusion, in Sections 4 and 5 respectively, we carry out the induction arguments on g (for $n = 3$) and on n, thereby showing that the candidate branch data $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ with some $\ell(\pi_j) = 2$ and not listed in Theorem 3.1 are indeed realizable.

1 Dessins d'enfant and topological reduction moves

As already mentioned, our proof of Theorem 3.1 for candidate branch data $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_{n-1}, [\![s, d-s]\!])$ relies on two induction arguments, one on g for $n = 3$, and then one on n. The core ingredient of both arguments is the following notion:

Definition 1.1. Let $\mathcal{D} = (g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ be a symbol involving generic integers g, d, n and partitions $\pi_1, \ldots, \pi_n$ of d subject to a set $\mathcal{C}$ of conditions. Let $\mathcal{D}' = (g' \cdot T, S, d', n'; \pi'_1, \ldots, \pi'_{n'})$ be a similar symbol, where $g', d', n', \pi'_1, \ldots, \pi'_{n'}$ depend on $g, d, n, \pi_1, \ldots, \pi_n$. We say that there is a *reduction move* $\mathcal{D} \rightsquigarrow \mathcal{D}'$ subject to $\mathcal{C}$ if the following happens:

- Whenever $\mathcal{D}$ is a candidate branch datum and $g, d, n, \pi_1, \ldots, \pi_n$ satisfy $\mathcal{C}$ then $\mathcal{D}'$ is also a candidate branch datum;
- The realizability of $\mathcal{D}'$ implies that of $\mathcal{D}$.

To describe our reduction moves we add an extra bit of notation. For η, ρ partitions of integers p, q we define $\eta * \rho$ as the partition of $p + q$ obtained by juxtaposing η and ρ (recall that the order is immaterial). Moreover when writing a reduction move $\mathcal{D} \rightsquigarrow \mathcal{D}'$ we will highlight using a boldface character those elements of $g, d, n, \pi_1, \ldots, \pi_n$ in $\mathcal{D}$ and those of $g', d', n', \pi'_1, \ldots, \pi'_{n'}$ in $\mathcal{D}'$ that do not coincide with the corresponding elements of the other symbol.

Dessins d'enfant We review here a notion popularized by Grothendieck in [9] (see also [1] and the more general [13]), recalling its connection with the Hurwitz existence problem.

Definition 1.2. We call *dessin d'enfant* a finite graph Γ in a surface Σ such that:

- Γ is bipartite, namely each of its vertices is colored black or white;
- Each edge of Γ has a black and a white end;
- $\Sigma \setminus \Gamma$ is a union of topological open discs, called *regions*.

If R is a region, the *length* of R is half the number of edges of Γ adjacent to R, counted twice if R is incident from both sides.

Proposition 1.3. *A candidate branch datum $\mathcal{D} = (\Sigma, S, d, 3; \pi_1, \pi_2, \pi_3)$ is realizable if and only if there exists a dessin d'enfant in Σ with valences of the black (respectively, white) vertices given by the entries of π_1 (respectively, π_2), and lengths of the complementary regions given by the entries of π_3.*

The "only if" part of the statement is obtained by defining Γ as $f^{-1}(e)$, where f is a branched cover realizing $\mathcal{D}$ and e is a segment with ends p_1 and p_2 and avoiding p_3, where $p_j \in S$ is the branching point corresponding to π_j, with $f^{-1}(p_1)$ black and $f^{-1}(p_2)$ white. The "if" part is achieved by reversing this construction. Note that the length of a complementary region can also be defined as the number of black (or, equivalently, white) vertices adjacent to it, counted with multiplicity.

Collapse of segments We now describe a key ingredient underlying the reduction moves described in the rest of this section. Let Γ be a bipartite graph in a surface Σ and let c be a segment in Σ such that $\Gamma \cap c$ consists of two distinct vertices of Γ with the same colour. Then we can construct a new bipartite graph Γ' in Σ as that obtained from Γ by collapsing c to a point, thus fusing together

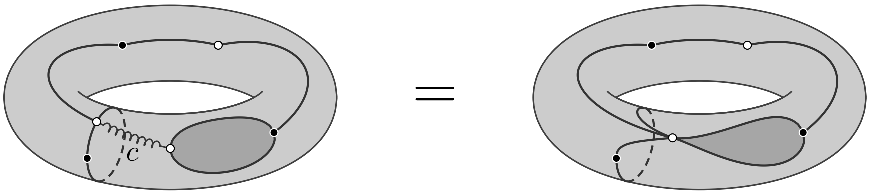

Figure 1: Collapse of a curly segment

its ends, as in Fig. 1. Note that the complementary regions of Γ' naturally correspond to those of Γ. This construction extends to the case where c is a finite family of segments $c_1, \ldots, c_k$ with pairwise disjoint interiors, by performing the collapses successively, provided the ends of each c_j stay distinct after $c_1, \ldots, c_{j-1}$ have been collapsed. Note that in our figures the segments to be collapsed are drawn as curly arcs, and that we will actually view figures with curly arcs as if these segments were already collapsed (whence the = sign in Fig. 1), so we will not employ any specific notation to express the fact that Γ' is a function of Γ and c.

Convention on figures A complementary region R of a dessin d'enfant in a surface Σ can be incident to itself along the boundary (in particular, its closure in Σ can fail to be a closed disc). However in our figures we will always unwind R, representing it as a closed disc with portions of Γ on its boundary only, as in the example of Fig. 2.

Trivial partitions Recall our convention that in a candidate branch datum $\mathcal{D} = (g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ all π_j's should be non-trivial. However, if a symbol $\mathcal{D}$ satisfies the Riemann-Hurwitz condition (1) and contains some $\pi_j = [\![1, \ldots, 1]\!]$, a candidate branch datum is obtained from $\mathcal{D}$ be removing these π_j's and correspondingly reducing n. We now have the following easy consequence of (1):

Remark 1.4. No candidate branch datum $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ exists with $\ell(\pi_n) = 2$ and $n \leqslant 2$.

Topological reduction moves We now state and prove the existence of reduction moves T_1, T_2, T_3, T_4 that we call *topological*, both because they allow to lower the genus of a candidate branch datum with $n = 3$, and because they are established using dessins d'enfant. We introduce these moves $T_j : \mathcal{D} \rightsquigarrow \mathcal{D}'$ in Propositions 1.5 to 1.8. In all four of them we leave to the reader the easy proof of the fact that if $\mathcal{D}$ is a candidate branch datum then $\mathcal{D}'$ also is. Note that by Remark 1.4 it is enough to show that if $\mathcal{D}$ satisfies (1) then $\mathcal{D}'$ also does.

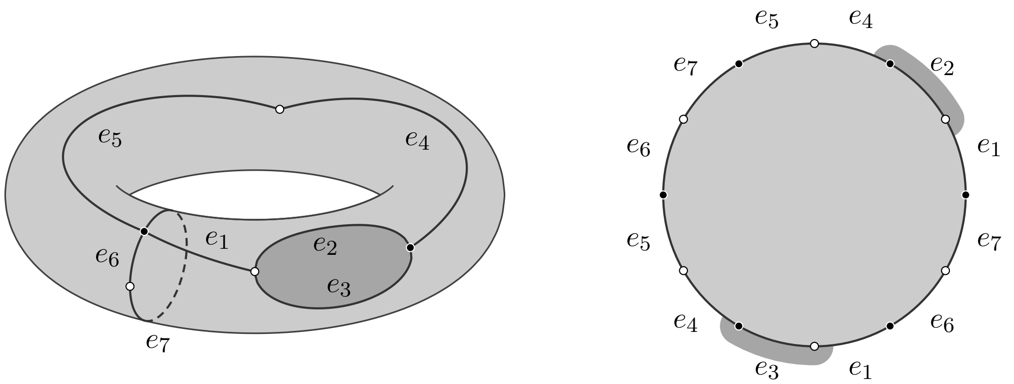

Figure 2: Left: a dessin d'enfant on the torus with a light grey and a dark grey complementary regions. Right: how to unwind the light grey region

Proposition 1.5. *For $g \geqslant 1$ the following is a reduction move:*

$$\begin{array}{rl} T_1: & \mathcal{D} = (\boldsymbol{g} \cdot T, S, d, 3; [\![1, 1, \mathbf{3}]\!] * \rho_1, \pi_2, [\![s, d-s]\!]) \\ \rightsquigarrow & \mathcal{D}' = ((\boldsymbol{g-1}) \cdot T, S, d, 3; [\![1, 1, \mathbf{1}, \mathbf{1}, \mathbf{1}]\!] * \rho_1, \pi_2, [\![s, d-s]\!]). \end{array}$$

Proof. Assume that $\mathcal{D}'$ is realizable and let Γ' realize it according to Proposition 1.3. Since π'_1 contains five 1's, at least three 1's are incident to least one complementary region of Γ', say the light grey one R, as in part 0 of Fig. 3. We then proceed as follows:

1. We attach to $(g-1) \cdot T$ a 1-handle with both attaching discs inside R (see part 1 of Fig. 3);

2. We collapse along two curly segments as in part 2 of Fig. 3.

The resulting bipartite graph Γ is a dessin d'enfant realizing $\mathcal{D}$. □

Proposition 1.6. *For $g \geqslant 1$, $x \geqslant 4$, $2 \leqslant s \leqslant d-2$, $x_1, x_2 \geqslant 1$ and $x_1 + x_2 = x - 2$ the following is a reduction move:*

$$\begin{array}{rl} T_2: & \mathcal{D} = (\boldsymbol{g} \cdot T, S, \boldsymbol{d}, 3; [\![\boldsymbol{x}]\!] * \rho_1, [\![\mathbf{2}]\!] * \rho_2, [\![\boldsymbol{s}, \boldsymbol{d-s}]\!]) \\ \rightsquigarrow & \mathcal{D}' = ((\boldsymbol{g-1}) \cdot T, S, \boldsymbol{d-2}, 3; [\![\boldsymbol{x_1}, \boldsymbol{x_2}]\!] * \rho_1, \rho_2, [\![\boldsymbol{s-1}, \boldsymbol{d-s-1}]\!]). \end{array}$$

Proof. Take a dessin d'enfant Γ' realizing $\mathcal{D}'$. We consider two cases, showing in both of them how to construct from Γ' a dessin d'enfant realizing $\mathcal{D}$.

Case 1: *The black vertex of valence x_1 is adjacent to one complementary region of Γ' and that of valence x_2 is adjacent to the other one.* We are in the situation of part 0 in Fig. 4, and we proceed as follows, always referring to Fig. 4:

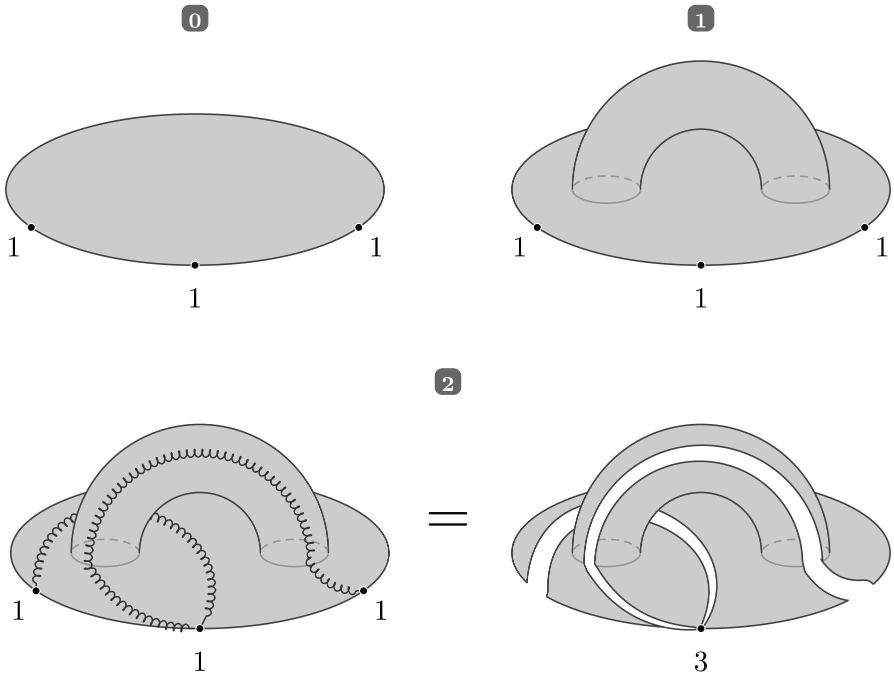

Figure 3: How to realize the reduction move T_1 at the level of dessins d'enfant. In part 2 we show for the last time both the curly arcs that must be viewed as already collapsed and the result of this collapse. We will not do this again in the next figures

1. We attach to $(g-1)\cdot T$ a 1-handle with attaching discs inside the two complementary regions, as in part 1;
2. We add to Γ' the co-core of the 1-handle, in the form of a black and a white vertex joined by two arcs, as in part 2;
3. We collapse along two curly segments, as in part 3.

CASE 2: *The two black vertices of valences x_1 and x_2 are completely surrounded by one of the complementary regions of* Γ' (the light grey one R, say). Note that there must be an edge e of R separating it from the other region, as in part 0 of Fig. 5. Then:

1. We add one black vertex and one white vertex on e (part 1);
2. We attach to $(g-1)\cdot T$ a 1-handle with attaching discs in R (part 2);
3. We collapse along two curly segments as in part 3. □

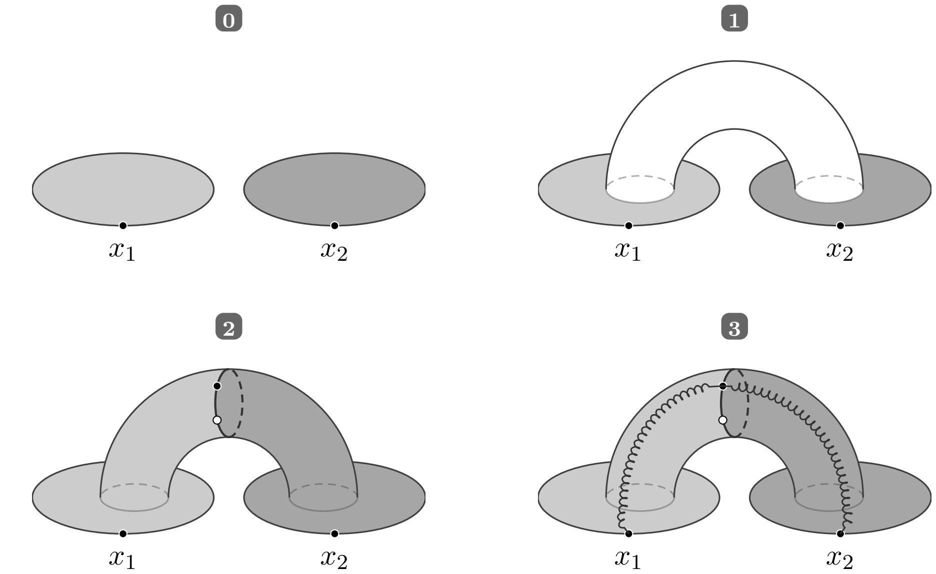

Figure 4: The move T_2 at the level of dessins d'enfant. Case 1

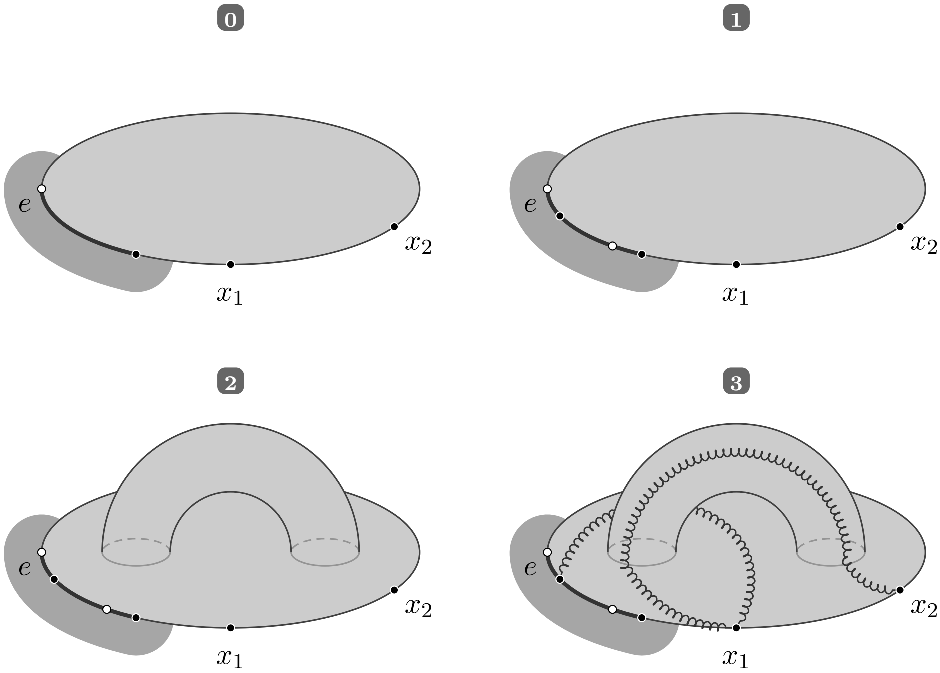

Figure 5: The move T_2 at the level of dessins d'enfant. Case 2

Proposition 1.7. *For $g \geqslant 1$, $x, y \geqslant 3$ and $3 \leqslant s \leqslant d-3$ the following is a reduction move:*

$$\begin{aligned} T_3: \quad & \mathcal{D} = (\boldsymbol{g} \cdot T, S, \boldsymbol{d}, 3; [\![\boldsymbol{x}, \boldsymbol{y}]\!] * \rho_1, [\![\mathbf{2}, \mathbf{2}]\!] * \rho_2, [\![\boldsymbol{s}, \boldsymbol{d}-\boldsymbol{s}]\!]) \\ \rightsquigarrow \quad & \mathcal{D}' = ((\boldsymbol{g}-\mathbf{1}) \cdot T, S, \boldsymbol{d}-\mathbf{4}, 3; [\![\boldsymbol{x}-\mathbf{2}, \boldsymbol{y}-\mathbf{2}]\!] * \rho_1, \rho_2, [\![\boldsymbol{s}-\mathbf{2}, \boldsymbol{d}-\boldsymbol{s}-\mathbf{2}]\!]). \end{aligned}$$

Proof. The structure of the proof is similar to the previous one.

CASE 1: *The black vertex of valence $x-2$ is adjacent to one complementary region of Γ' and that of valence $y-2$ is adjacent to the other one*, as in part 0 of Fig. 6. Then:

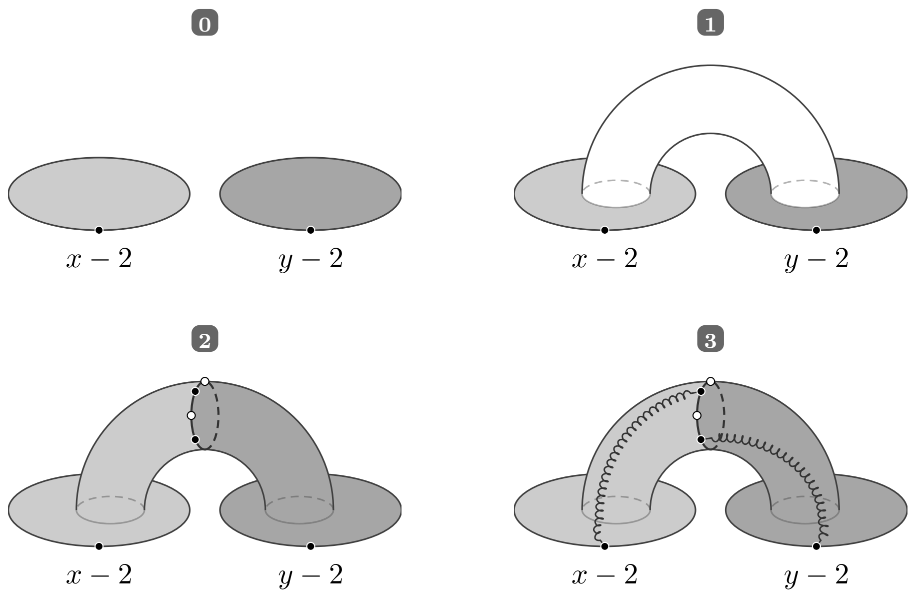

Figure 6: The move T_3 at the level of dessins d'enfant. Case 1

1. We attach to $(g-1) \cdot T$ a 1-handle with attaching discs inside the two complementary regions, as in part 1;
2. We add to Γ' the co-core of the 1-handle, in the form of two black and two white vertices joined by four arcs, as in part 2;
3. We collapse along two curly segments as in part 3.

CASE 2: *The two black vertices of valences $x-2$ and $y-2$ are completely surrounded by one of the complementary regions of Γ'* (the light grey one R, say). Take an edge e of R separating it from the other region, as in part 0 of Fig. 7. Then:

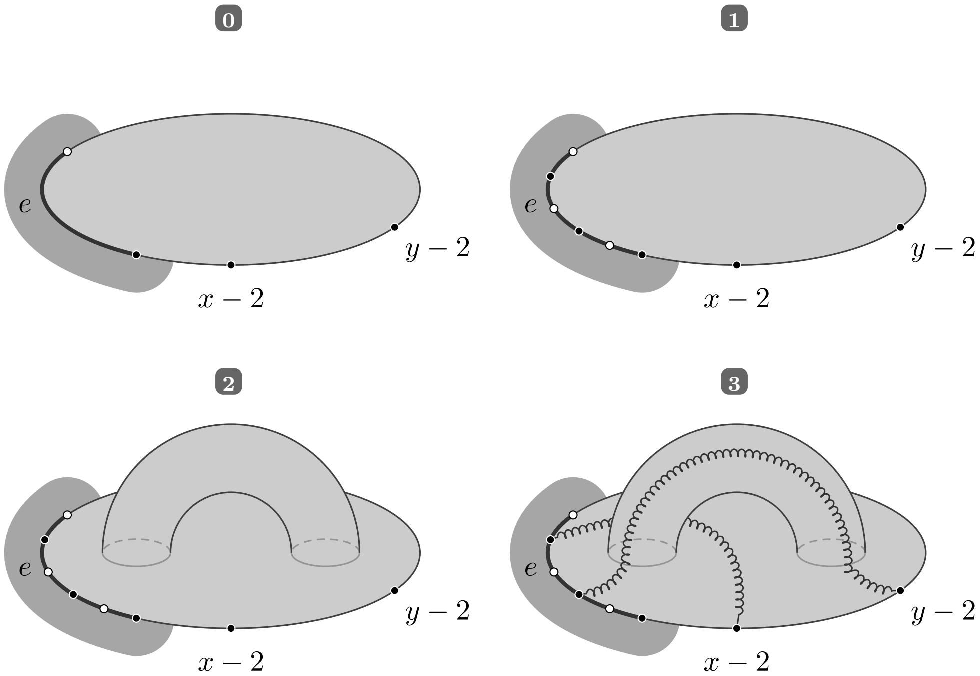

Figure 7: The move T_3 at the level of dessins d'enfant. Case 2

1. We add two black and two white vertices on e (part 1);
2. We attach to $(g-1)\cdot T$ a 1-handle with attaching discs in R (part 2);
3. We collapse along two curly segments as in part 3. □

Proposition 1.8. *For $g \geqslant 1$, $x \geqslant 4$, $y \geqslant 3$ and $2 \leqslant s \leqslant d-2$ the following is a reduction move:*

$$\begin{aligned} T_4: \quad & \mathcal{D} = (\boldsymbol{g}\cdot T, S, \boldsymbol{d}, 3; [\![\boldsymbol{x}]\!] * \rho_1, [\![\boldsymbol{y}]\!] * \rho_2, [\![\boldsymbol{s}, \boldsymbol{d}-\boldsymbol{s}]\!]) \\ \rightsquigarrow \quad & \mathcal{D}' = ((\boldsymbol{g}-\boldsymbol{1})\cdot T, S, \boldsymbol{d}-\boldsymbol{2}, 3; [\![\boldsymbol{x}-\boldsymbol{2}]\!] * \rho_1, [\![\boldsymbol{y}-\boldsymbol{2}]\!] * \rho_2, [\![\boldsymbol{s}-\boldsymbol{1}, \boldsymbol{d}-\boldsymbol{s}-\boldsymbol{1}]\!]). \end{aligned}$$

Proof. The argument is again similar to those showing Propositions 1.6 and 1.7, starting with a dessin d'enfant Γ' realizing $\mathcal{D}'$, with two cases to consider. But in case 2 we will possibly have to change the given Γ' before acting on it, so the explanation is longer. We call u the black vertex of Γ' of valence $x-2$, and v the white one of valence $y-2$.

CASE 1: *u is adjacent to one complementary region of Γ' and v is adjacent to the other one*, as in part 0 of Fig. 8. Then:

1. We attach to $(g-1)\cdot T$ a 1-handle with attaching discs inside the two complementary regions, as in part 1;

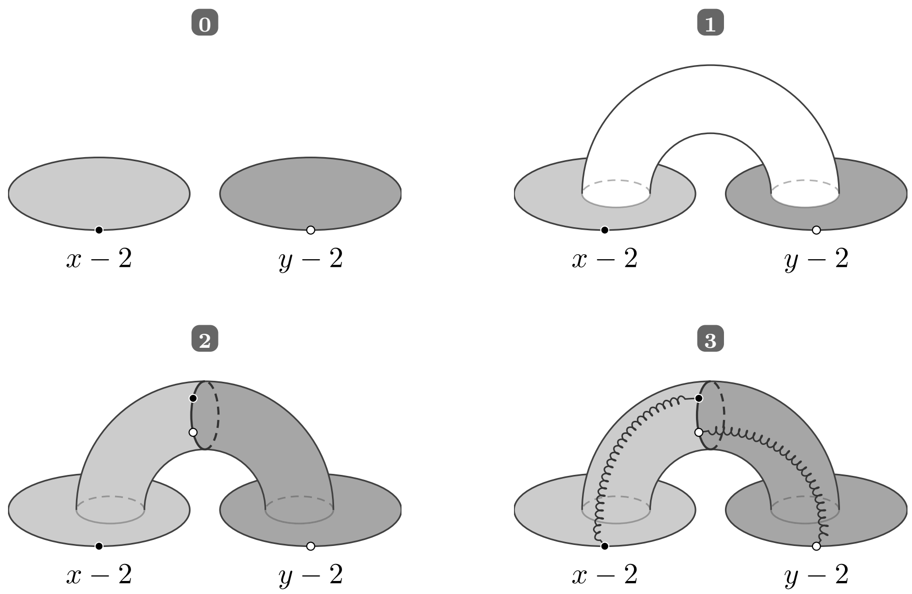

Figure 8: The move T_4 at the level of dessins d'enfant. Case 1

2. We add to Γ' the co-core of the 1-handle, in the form of a black and a white vertex joined by two arcs, as in part 2;

3. We collapse along two curly segments as in part 3.

Case 2: *u and v are completely surrounded by one of the complementary regions of* Γ' (the light grey one R, say). We claim that up to changing Γ' we can realize the following situation: *There is an edge e shared by the two complementary regions of Γ' such that along ∂R (for one of the two possible orientations) we see (perhaps not consecutively) u, then e with its white end first and its black end last, and then v*, as in Fig. 9. Note that along ∂R (after we unwind it) we see u

Figure 9: The configuration on ∂R that we aim to realize

exactly $x-2 \geqslant 2$ times and v exactly $y-2 \geqslant 1$ times. If there is a shared edge e such that u, v, u, e appear in this order (perhaps not consecutively) on ∂R, as in Fig. 10 then of course we have the desired configuration.

Otherwise, we have to modify Γ'. We proceed as follows, referring to Fig. 11.

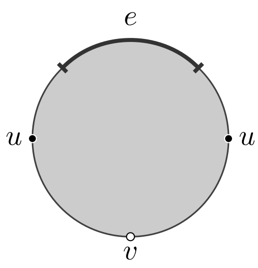

Figure 10: A configuration that guarantees that of Fig. 9

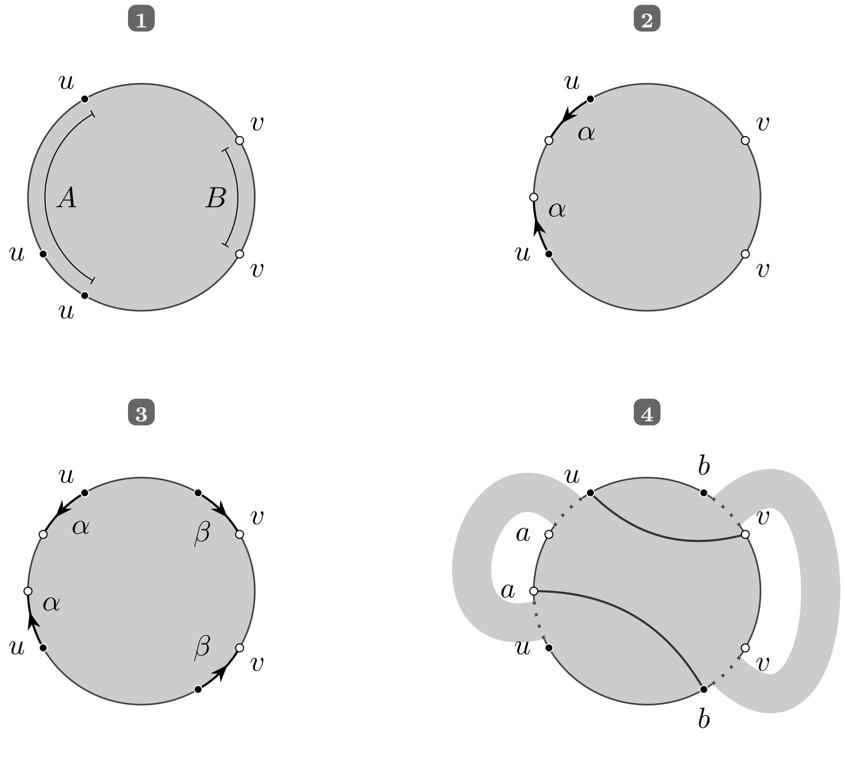

Figure 11: How to modify Γ' to get a configuration as in Fig. 10

1. If no u, v, u, e configuration appears along ∂R there is an arc A of ∂R that contains all the occurrences of u, and none of v or any shared edge. Similarly, there is an arc B of ∂R that contains all the occurrences of v, none of u and no shared edge. Note that u and v are not the ends of a shared edge, since they are completely surrounded by R.

2. Choose an orientation of ∂R (counterclockwise in the picture), take the first occurrence of u in A and let α be the edge immediately after it. Since α is not shared, it appears again on ∂R, with the opposite orientation. Note that α cannot occur immediately before the first appearance of u, otherwise u would have valence 1, therefore it will occur somewhere else on A.

3. Take the first occurrence of v in B and let β be the edge immediately before in ∂R. Since β is not shared, it will also occur elsewhere on ∂R.

4. Let a be the end of α other than u and b be end of β other than v. Erase the edges α and β and draw two new ones, connecting u to v and a to b as shown in the picture, getting a new dessin d'enfant Γ'. One easily sees that Γ' still realizes $\mathcal{D}'$, and now on the boundary of the light grey complementary region we see u, v, u in this order, without shared edges in between (because there was none on B). But some shared edge exists, so we are in the situation already examined.

Our claim is proved and we can conclude as follows (see Fig. 12):

1. We add a black and a white vertex on e (part 1);

2. We attach to $(g-1) \cdot T$ a 1-handle with attaching discs in R (part 2);

3. We collapse along two curly segments as in part 3. □

2 Monodromy and algebraic reduction moves

We recall here the monodromy reformulation of the Hurwitz existence problem and some results from [5], deducing the existence of two more reduction moves. In this section we introduce some notation not used in the rest of the paper except within the proof of Theorem 5.2. For $d \geqslant 2$ we denote by Π_d the set of partitions of d, and for $\pi \in \Pi_d$ we set $v(\pi) = d - \ell(\pi)$, where $\ell(\pi)$ is the length of π. Using v, the Riemann-Hurwitz condition (1) for a candidate branch datum $(\Theta, \Sigma, d, n; \pi_1, \ldots, \pi_n)$ can be written as

$$d\chi(\Sigma) - \chi(\Theta) = v(\pi_1) + \ldots + v(\pi_n),$$

hence for $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ as

$$v(\pi_1) + \ldots + v(\pi_n) = 2(d + g - 1). \qquad (2)$$

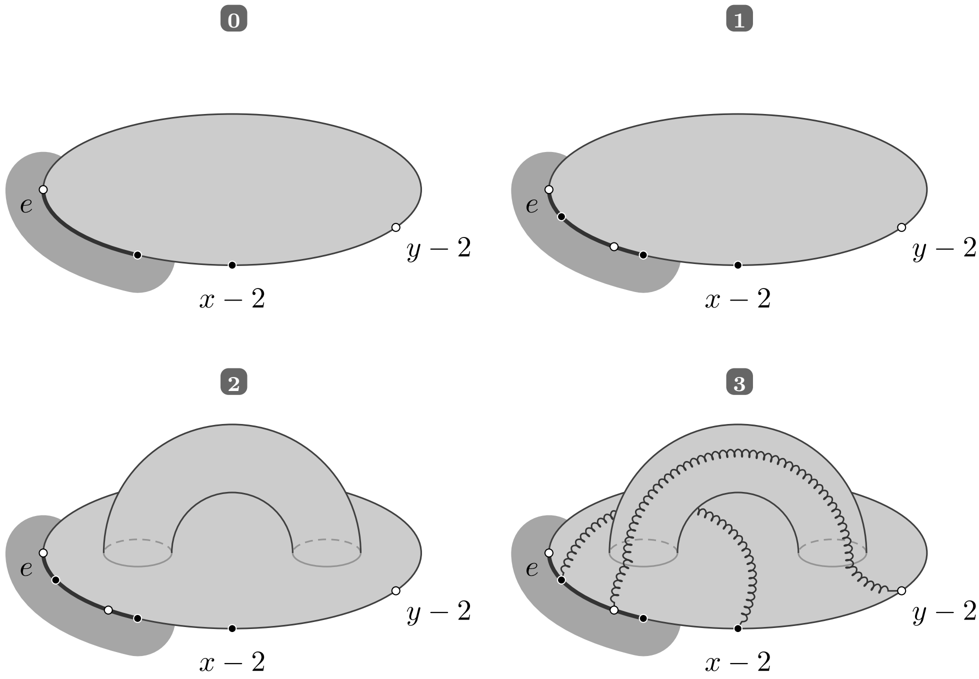

Figure 12: The move T_4 at the level of dessins d'enfant. Case 2

For $\alpha \in \mathfrak{S}_d$, the group of permutations of $\{1, \ldots, d\}$, we denote by $\pi(\alpha) \in \Pi_d$ the partition of d given by the lengths of the cycles of α (including the trivial ones of length 1) and we set $\ell(\alpha) = \ell(\pi(\alpha))$ and $v(\alpha) = v(\pi(\alpha))$.

Review of known results We begin by citing a result of [10] and deducing one from [5]:

Proposition 2.1. *A candidate branch datum* $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ *is realizable if and only if there exist* $\theta_1, \ldots, \theta_n \in \mathfrak{S}_d$ *such that:*

1. $\pi(\theta_i) = \pi_i$ *for* $i = 1, \ldots, n$*;*
2. $\theta_1 \cdots \theta_n = \mathrm{id}$*;*
3. *The subgroup* $\langle \theta_1, \ldots, \theta_n \rangle$ *of* $\mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$*.*

Proposition 2.2. *If* $\pi_1, \pi_2 \in \Pi_d$ *and* $v(\pi_1) + v(\pi_2) \leqslant d - 1$ *there exist* $\theta_1, \theta_2 \in \mathfrak{S}_d$ *such that* $\pi(\theta_i) = \pi_i$ *for* $i = 1, 2$ *and* $v(\theta_1 \cdot \theta_2) = v(\pi_1) + v(\pi_2)$*.*

Proof. According to Lemma 4.2 in [5], if $v(\pi_1) + v(\pi_2) = d - t$ then there exist $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$, the subgroup $\langle \theta_1, \theta_2 \rangle$ of $\mathfrak{S}_d$ has precisely t orbits, and $\pi(\theta_1 \cdot \theta_2)$ is the partition of d given by the lengths of these orbits. This implies that $v(\theta_1 \cdot \theta_2) = d - t$, so $v(\theta_1 \cdot \theta_2) = v(\pi_1) + v(\pi_2)$. □

We then reformulate Corollary 4.4 and Lemma 4.5 from [5], respectively:

Proposition 2.3. *Let $\pi_1, \pi_2 \in \Pi_d$ be such that $v(\pi_1) + v(\pi_2) \geqslant d - 1$ and $v(\pi_1) + v(\pi_2) \equiv d - 1 \pmod 2$. Then there exist $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$ and $\pi(\theta_1 \cdot \theta_2) = [\![d]\!]$.*

Proposition 2.4. *Let $\pi_1, \pi_2 \in \Pi_d$ be such that $v(\pi_1) + v(\pi_2) \geqslant d$ and $v(\pi_1) + v(\pi_2) \equiv d \pmod 2$. Then there exist $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$ and*

$$\pi(\theta_1 \cdot \theta_2) = \begin{cases} [\![d/2, d/2]\!] & \text{if} \quad \pi_1 = \pi_2 = [\![2, \ldots, 2]\!] \\ [\![d-1, 1]\!] & \text{otherwise.} \end{cases}$$

Algebraic reduction moves We introduce here two moves A_1 and A_2 used in Section 5.

Proposition 2.5. *If $\pi_1, \pi_2 \in \Pi_d$ and $v(\pi_1) + v(\pi_2) \leqslant d - 1$, for all $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$ and $v(\theta_1 \cdot \theta_2) = v(\pi_1) + v(\pi_2)$, setting $\pi = \pi(\theta_1 \cdot \theta_2)$ we have that the following is a reduction move:*

$$\begin{array}{rl} A_1 : & \mathcal{D} = (g \cdot T, S, d, \boldsymbol{n}; \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \pi_3, \ldots, \pi_n) \\ \rightsquigarrow & \mathcal{D}' = (g \cdot T, S, d, \boldsymbol{n-1}; \boldsymbol{\pi}, \pi_3, \ldots, \pi_n). \end{array}$$

Proof. Version (2) of the Riemann-Hurwitz relation implies that if $\mathcal{D}$ as in the statement is a candidate branch datum, $\mathcal{D}'$ also is (since $v(\pi) = v(\pi_1) + v(\pi_2) > 0$ we see that π is non-trivial). Now if $\theta, \theta_3, \ldots, \theta_n$ realize $\mathcal{D}'$ according to Proposition 2.1, we can assume that $\theta = \theta_1 \cdot \theta_2$, whence $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ realize $\mathcal{D}$. □

Proposition 2.6. *Given $\pi_1, \ldots, \pi_n \in \Pi_d$ with $v(\pi_1) + v(\pi_2) \geqslant d - 1$ and $v(\pi_3) + \ldots + v(\pi_n) \geqslant d - 1$, if $g = \frac{1}{2}(v(\pi_1) + \ldots + v(\pi_n)) - d + 1$ is a non-negative integer then there exists $g' \in \mathbb{N}$ and $\pi \in \Pi_d$ such that the following is a reduction move:*

$$\begin{array}{rl} A_2 : & \mathcal{D} = (\boldsymbol{g} \cdot T, S, d, \boldsymbol{n}; \boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \pi_3, \ldots, \pi_n) \\ \rightsquigarrow & \mathcal{D}' = (\boldsymbol{g'} \cdot T, S, d, \boldsymbol{n-1}; \boldsymbol{\pi}, \pi_3, \ldots, \pi_n), \end{array}$$

where $\pi = [\![d/2, d/2]\!]$ if $\pi_1 = \pi_2 = [\![2, \ldots, 2]\!]$, otherwise

$$\pi = \begin{cases} [\![d]\!] & \text{if} \quad v(\pi_1) + v(\pi_2) \equiv d - 1 \pmod 2 \\ [\![d-1, 1]\!] & \text{if} \quad v(\pi_1) + v(\pi_2) \equiv d \pmod 2. \end{cases}$$

Proof. Set $v_j = v(\pi_j)$. If $v_1 + v_2 \equiv d - 1 \pmod 2$ we can apply Proposition 2.3 getting $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$ and $\pi(\theta_1 \cdot \theta_2) = [\![d]\!]$. If instead $v_1 + v_2 \equiv d \pmod 2$ we note that $v_1 + v_2 \geqslant d - 1$ implies $v_1 + v_2 \geqslant d$, so we can apply Proposition 2.4, getting $\theta_1, \theta_2 \in \mathfrak{S}_d$ such that $\pi(\theta_i) = \pi_i$ for $i = 1, 2$ and

$$\pi(\theta_1 \cdot \theta_2) = \begin{cases} [\![d/2, d/2]\!] & \text{if} \quad \pi_1 = \pi_2 = [\![2, \ldots, 2]\!] \\ [\![d-1, 1]\!] & \text{otherwise.} \end{cases}$$

Setting $\pi = \pi(\theta_1 \cdot \theta_2)$ we claim that in both cases if $\mathcal{D}$ as in the statement is a candidate branch datum, so is $\mathcal{D}'$ for a suitable $g' \in \mathbb{N}$. Once this is done, the conclusion is precisely as in the previous proof. First of all, π is always non-trivial. Next, we must show that (2) holds for $\mathcal{D}'$, namely that

$$v(\pi) + v_3 + \ldots + v_n = 2(d + g' - 1)$$

holds for some $g' \in \mathbb{N}$, or equivalently that

$$z = v(\pi) + v_3 + \ldots + v_n - 2d + 2$$

is even and non-negative. Now in each of the three cases one readily sees that $v(\pi) \equiv v_1 + v_2 \pmod 2$, but $v_1 + v_2 + v_3 + \ldots + v_n$ is even by (2) for $\mathcal{D}$, so indeed z is even. Moreover $v(\pi) \geqslant d - 2$ and $v_3 + \ldots + v_n \geqslant d - 1$, so $z \geqslant -1$ and the conclusion follows. □

3 Statement and exceptionality

We now state the result to which the present paper is devoted:

Theorem 3.1. *A candidate branch datum $(g \cdot T, S, d, n; \pi_1, \ldots, \pi_n)$ with $\ell(\pi_n) = 2$ is exceptional if and only if it is one of the following:*

(1) $(S, S, 12, 3; [\![2,2,2,2,2,2]\!], [\![1,1,1,3,3,3]\!], [\![6,6]\!])$;

(2) $(S, S, 2k, 3; [\![2, \ldots, 2]\!], [\![2, \ldots, 2]\!], [\![s, 2k-s]\!])$ *with* $k \geqslant 2$ *and* $s \neq k$;

(3) $(S, S, 2k, 3; [\![2, \ldots, 2]\!], [\![1, 2, \ldots, 2, 3]\!], [\![k, k]\!])$ *with* $k \geqslant 2$;

(4) $(S, S, 4k+2, 3; [\![2, \ldots, 2]\!], [\![1, \ldots, 1, k+1, k+2]\!], [\![2k+1, 2k+1]\!])$ *with* $k \geqslant 1$;

(5) $(S, S, 4k, 3; [\![2, \ldots, 2]\!], [\![1, \ldots, 1, k+1, k+1]\!], [\![2k-1, 2k+1]\!])$ *with* $k \geqslant 2$;

(6) $(S, S, kh, 3; [\![h, \ldots, h]\!], [\![1, \ldots, 1, k+1]\!], [\![ph, (k-p)h]\!])$ *with* $h \geqslant 2$, $k \geqslant 2$ *and* $0 < p < k$;

(7) $(T, S, 6, 3; [\![3,3]\!], [\![3,3]\!], [\![2,4]\!])$;

(8) $(T, S, 8, 3; [\![2,2,2,2]\!], [\![4,4]\!], [\![3,5]\!])$;

(9) $(T, S, 12, 3; [\![2,2,2,2,2,2]\!], [\![3,3,3,3]\!], [\![5,7]\!])$;

(10) $(T, S, 16, 3; [\![2,2,2,2,2,2,2,2]\!], [\![1,3,3,3,3,3]\!], [\![8,8]\!])$;

(11) $(T, S, 2k, 3; [\![2,\ldots,2]\!], [\![2,\ldots,2,3,5]\!], [\![k,k]\!])$ *with* $k \geqslant 5$*;*

(12) $(2 \cdot T, S, 8, 4; [\![2,2,2,2]\!], [\![2,2,2,2]\!], [\![2,2,2,2]\!], [\![3,5]\!])$*;*

(13) $((n-3) \cdot T, S, 4, n; [\![2,2]\!], \ldots, [\![2,2]\!], [\![1,3]\!])$ *with* $n \geqslant 3$.

Remark 3.2. By Remark 1.4 we always assume from now on that $n \geqslant 3$.

Remark 3.3. For the case where the candidate covering surface is the sphere S, Pakovich [20] lists 7 exceptional families rather than the 6 above. Our statement is however coherent with his one, because two of his families actually coincide. In fact, items (4) and (5) in his statement, using our current notation, are respectively

(4) $(S, S, d, 3; [\![2,\ldots,2]\!], [\![1,\ldots,1,q,q]\!], [\![2q-3, d-2q+3]\!])$ with $q \geqslant 3$;

(5) $(S, S, d, 3; [\![2,\ldots,2]\!], [\![1,\ldots,1,q,q]\!], [\![2q-1, d-2q+1]\!])$ with $q \geqslant 3$.

Of course in both cases we must have $d = 2t$, and the Riemann-Hurwitz condition (1) reads

$$2 - (t + (2t - 2q + 2) + 2) = 2t(2-3) \quad \Rightarrow \quad t = 2q - 2.$$

We can then define k as $q - 1$ and we have

$$k \geqslant 2, \qquad q = k+1, \quad t = 2k, \quad d = 4k.$$

Moreover

$$\begin{cases} 2q-3 = 2k+2-3 = 2k-1 \\ d-(2q-3) = 2k+1 \end{cases} \qquad \begin{cases} 2q-1 = 2k+2-1 = 2k+1 \\ d-(2q-1) = 2k-1 \end{cases}$$
$$\Rightarrow \quad [\![2q-3, d-2q+3]\!] = [\![2q-1, d-2q+1]\!] = [\![2k-1, 2k+1]\!]$$

so both candidate branch data coincide with our (5).

Remark 3.4. The items in our statements have a few overlaps. For instance item (2) with $k = 2$ coincides with item (6) with $k = h = 2$. Easy extra restrictions on the parameters appearing would lead to a list without overlaps, but we will not make this explicit.

In the rest of the section we prove that items (1) to (13) in Theorem 3.1 are indeed exceptional.

Exceptionality with covering surface the sphere We begin with items (1) to (6) in Theorem 3.1 (whose exceptionality was taken for granted in [20]). For item (1), one could use the data of [33], but a proof via dessins d'enfant is also very easy:

Proposition 3.5. *The candidate branch datum (1) in Theorem 3.1 is exceptional.*

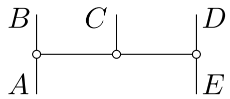

Figure 13: A graph in S

Proof. A dessin d'enfant Γ realizing (1), ignoring the 2-valent black vertices, would be a connected graph in S with three 3-valent and three 1-valent white vertices, and 6 edges. Then Γ is obtained from the graph of Fig. 13 by joining two of its free ends A, B, C, D, E and putting a valence-1 vertex at the end of the other three. Of the 10 possible junctions, up to symmetry we can consider only A-B, A-C, A-D, A-E, B-C, B-D, that give for the lengths of the complementary regions respectively $[\![1,11]\!]$, $[\![4,8]\!]$, $[\![5,7]\!]$, $[\![3,9]\!]$, $[\![2,10]\!]$, $[\![5,7]\!]$, so $[\![6,6]\!]$ does not appear. □

Item (2) is exceptional because a dessin d'enfant realizing it would be a circle with $2k$ valence-2 vertices on it (of alternating colours), but then its complementary regions would have lengths $[\![k,k]\!]$. Exceptionality of item (3) was shown in [23, Proposition 1.3]. That of items (4) to (6) perhaps follows from some published result, but showing it via dessins d'enfant is quite easy, so we do it, starting from the hardest case (6) and then providing sketches for (4) and (5). We note that item (6) for the special case $p = 1$ is treated in [5, Proposition 5.7].

Proposition 3.6. *Item (6) in Theorem 3.1 is exceptional.*

Proof. A dessin d'enfant $\Gamma \subset S$ realizing the datum, being connected, must appear as in Fig. 14 for some $0 \leqslant a \leqslant k-1$ and $0 \leqslant b \leqslant h-2$. But then one sees that one of the complementary regions has length $ah+b+1$ which cannot be ph or $(k-p)h$ because it is not a multiple of h. □

Proposition 3.7. *Items (4) and (5) in Theorem 3.1 are exceptional.*

Proof. Suppose a dessin d'enfant Γ realizes one of them. Then, neglecting the valence-2 black vertices, we see that Γ is obtained from a segment with white ends A and B by attaching another segment with ends either both at A or one at A and one at B, and some other segments with a white valence-1 end and the other at A or B. Note that one has to take into account in which of the two regions created by the first two segments these last segments are added. However, if the second segment has one end at A and one at B, the realized π_3 has even entries, so to prove exceptionality of (4) and (5) we can assume the second segment has both ends at A. Now, if a extra segments are added to A in the "small" complementary region then the realized π_3 is $[\![2a+1, d-2a-1]\!]$. For item (4) we note that A has valence at most $k+2$, whence $a \leqslant k-1$, so $2a+1 \leqslant 2k-1$ is never $2k+1$. Similarly, for item (5) we note that A has valence $k+1$, whence $a \leqslant k-2$, so $2a+1 \leqslant 2k-3$ is never $2k-1$ or $2k+1$. □

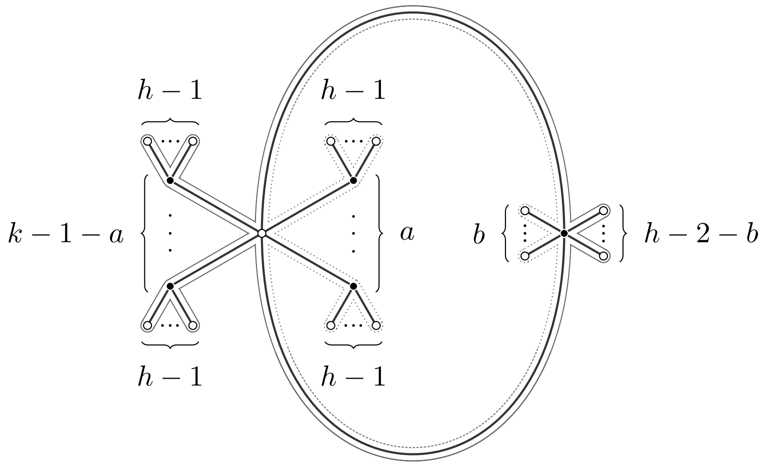

Figure 14: Proof of the exceptionality of item (6)

Exceptionality with covering surface of positive genus As above for item (1), items (7) to (10) in Theorem 3.1 are very easily shown to be exceptional via dessins d'enfant, but we do not exhibit a proof as all these cases fall within the experimental analysis of Zheng [33]. Exceptionality of item (11) was shown in [23, Proposition 1.2].

Proposition 3.8. *The candidate branch datum (12) in Theorem 3.1 is exceptional.*

Proof. Again we could just apply the method of Zheng [33], but we sketch two alternative arguments.

SKETCHED ALTERNATIVE ARGUMENT 1 Suppose a branched covering f realizing item (12) exists, and let e be a segment with ends at the first two branching points, midpoint at the third one and avoiding the fourth one. Then $f^{-1}(e)$ is a graph Γ in $2 \cdot T$ with four 4-valent vertices and two complementary discs incident to respectively 6 and 10 vertices. The fact that no such Γ exists is shown as follows:

- Note that abstractly such a Γ always has as a maximal tree Λ as in Fig. 15 (if it has none then there is one that is a spider with a head A and three legs with ends B, C, D, but B cannot be joined to C or D, hence C is joined to D, a contradiction);

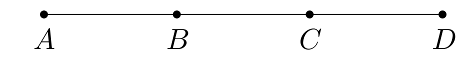

Figure 15: A tree Λ

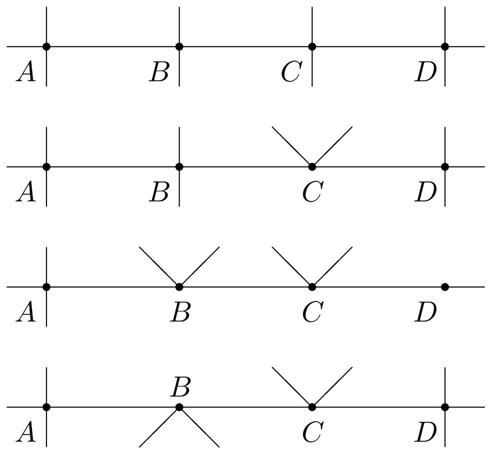

Figure 16: Possibilities for the planar neighbourhood U of Λ in Σ

- List all the ways the other 5 edges of Γ can be abstractly attached to Λ and eliminate duplicates up to symmetry, concluding that there are 10 possibilities for the abstract Γ (we do not show them explicitly);
- Note that if Γ is embedded in a surface Σ then a neighbourhood U of Λ in Σ is contained in a plane and it is one of the four shown in Fig. 16;
- Fix one of the 10 abstract Γ's and one of the U's of Fig. 16; now the compatible Γ's in Σ are those obtained by:
  - Joining in pairs, by arcs in the plane that may cross each other, the 10 germs of edges of Λ, so to get Γ;
  - Taking the circles bounding a neighbourhood of Γ in the plane (ignoring the crossings);
  - Attaching a disc to each such circle;
  - Computing to how many vertices these discs are incident;
- This description suggests that several cases have to be considered for each of the 10×4 possibilities. But our aim is just to show that we never find two attaching discs incident to respectively 6 and 10 vertices, so we can discard a partially constructed Γ as soon as we see it creates an attaching disc incident to some number of vertices different from 6 and 10. The

resulting analysis is then rather simple and quick, leading to the desired conclusion.

Sketched alternative argument 2 By Proposition 2.1 we must show that if $\theta_1, \theta_2, \theta_3 \in \mathfrak{S}_8$ have $\pi(\theta_j) = [\![2,2,2,2]\!]$ we never have $\pi(\theta_1 \cdot \theta_2 \cdot \theta_3) = [\![3,5]\!]$. Assuming $\theta_1 = (1,2)(3,4)(5,6)(7,8)$ and considering how many transpositions θ_2 shares with θ_1, we see that either $\theta_2 = \theta_1$, in which case the conclusion is obvious, or θ_2 can be taken to be one of

$$\begin{array}{ll}(1,2)(3,4),(5,7),(6,8) & (1,2)(3,5)(4,7)(6,8) \\ (1,3)(2,4),(5,7),(6,8) & (1,3)(2,5)(4,7)(6,8).\end{array}$$

Now there are $7 \cdot 5 \cdot 3$ choices for θ_3, so $4 \cdot 7 \cdot 5 \cdot 3 = 420$ cases to examine, which requires patience but can be done. □

Finally, item (13) was shown to be exceptional in [5], exploiting the easy fact that the identity and the elements σ of $\mathfrak{S}_4$ with $\pi(\sigma) = [\![2,2]\!]$ form a subgroup of $\mathfrak{S}_4$.

4 Realizability for three branching points

In this section we prove Theorem 3.1 under the restriction that the number n of branching points of the candidate branch datum is 3. As announced, we proceed by induction on the genus g of the candidate covering surface, with the base step $g = 0$ being a consequence of [20]:

Theorem 4.1. *The only exceptional candidate branch data of the form*

$$\mathcal{D} = (S, S, d, 3; \pi_1, \pi_2, [\![s, d-s]\!])$$

are items (1) to (6) in Theorem 3.1.

The inductive step will use the reduction moves $T_1, \ldots, T_4$ of Section 1.

Sparse realizability results For some specific candidate branch data we will not be able to apply any reduction move, so we treat them here. We begin by stating a fact which is easily deduced from [24], and then we employ dessins d'enfant to establish another fact.

Proposition 4.2. *For $x \geqslant 3$ and $y \geqslant 2$ any candidate branch datum as follows is realizable:*

$$\mathcal{D} = (g \cdot T, S, d, 3; [\![x,y]\!] * \rho_1, [\![2,2]\!] * \rho_2, [\![2, d-2]\!]).$$

Proposition 4.3. *For $g \geqslant 2$ the following candidate branch data are realizable:*

1. *$(g \cdot T, S, 6g, 3; [\![3, \ldots, 3]\!], [\![3, \ldots, 3]\!], [\![s, 6g-s]\!])$;*

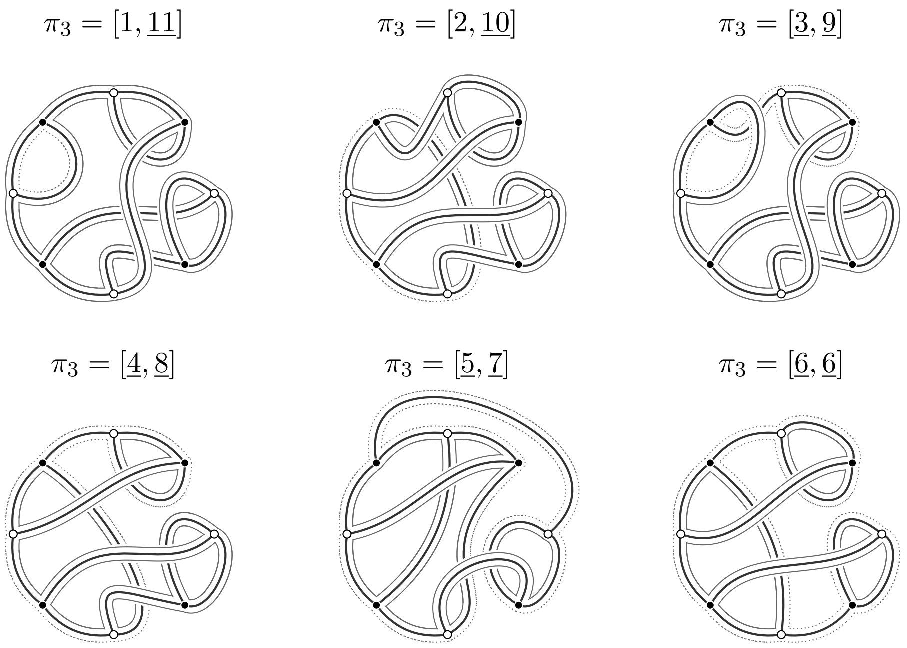

Figure 17: Fact 1 at the level of dessins d'enfant. These pictures only show an embedding in $\mathbb{R}^3$ of a regular neighbourhood of a dessin Γ in $2 \cdot T$

2. $(g \cdot T, S, 6g+2, 3; \llbracket 2, 3, \ldots, 3 \rrbracket, \llbracket 2, 3 \ldots, 3 \rrbracket, \llbracket s, 6g+2-s \rrbracket)$;
3. $(g \cdot T, S, 6g+3, 3; \llbracket 3, \ldots, 3 \rrbracket, \llbracket 1, 2, 3, \ldots, 3 \rrbracket, \llbracket s, 6g+3-s \rrbracket)$;
4. $(g \cdot T, S, 6g+4, 3; \llbracket 1, 3, \ldots, 3 \rrbracket, \llbracket 1, 3, \ldots, 3 \rrbracket, \llbracket s, 6g+4-s \rrbracket)$;
5. $(g \cdot T, S, 6g+6, 3; \llbracket 1, 2, 3, \ldots, 3 \rrbracket, \llbracket 1, 2, 3, \ldots, 3 \rrbracket, \llbracket s, 6g+6-s \rrbracket)$.

Proof. Within this proof we define an *augmented datum* as a symbol $\mathcal{D} = (g \cdot T, S, d, 3; \pi_1, \pi_2, \pi_3)$ similar to a candidate branch datum, except that some entries of π_3 are underlined. We say that $\mathcal{D}$ is *realizable* if there exists a dessin d'enfant Γ realizing $\mathcal{D}$ (forgetting the underlining) such that, for every complementary region R of Γ corresponding to an underlined entry of π_3, there is an edge of Γ with R on both sides.

Fact 1. *For* $\pi_3 \in \{\llbracket 1, \underline{11} \rrbracket, \llbracket 2, \underline{10} \rrbracket, \llbracket \underline{3}, \underline{9} \rrbracket, \llbracket \underline{4}, \underline{8} \rrbracket, \llbracket \underline{5}, \underline{7} \rrbracket, \llbracket \underline{6}, \underline{6} \rrbracket\}$ *the augmented datum* $\mathcal{D} = (2 \cdot T, S, 12, 3; \llbracket 3, 3, 3, 3 \rrbracket, \llbracket 3, 3, 3, 3 \rrbracket, \pi_3)$ *is realizable.* This is established by exhibiting the desired dessins d'enfant in Fig. 17.

Fact 2. *If* $\mathcal{D} = (g \cdot T, d, \pi_1, \pi_2, \llbracket \underline{x}, d-x \rrbracket)$ *is realizable then*

$$\mathcal{D}' = ((g+1) \cdot T, d+6, \llbracket 3, 3 \rrbracket * \pi_1, \llbracket 3, 3 \rrbracket * \pi_2, \llbracket \underline{x+6}, d-x \rrbracket)$$

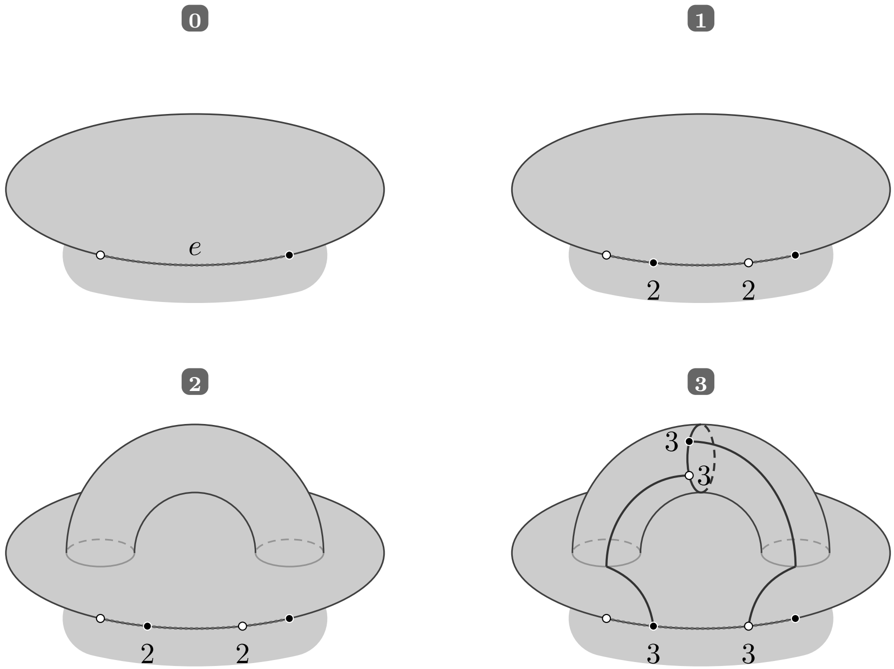

Figure 18: Fact 2 at the level of dessins d'enfant

also is. To see this, take a dessin d'enfant Γ realizing $\mathcal{D}$, with light grey complementary region R corresponding to $\underline{x}$, and fix an edge e of Γ with R on both sides, as in part 0 of Fig. 18. We then operate as follows on Γ to get a Γ' realizing $\mathcal{D}'$:

1. We add a black and a white vertex on e (part 1);
2. We attach to $g\cdot T$ a 1-handle with attaching discs inside R (part 2);
3. We add one black vertex, one white vertex and four edges, as in part 3.

Fact 3. *For* $g\geqslant 2$ *and* $\pi_3\in\{[\![1,\underline{6g-1}]\!],[\![2,\underline{6g-2}]\!]\}\cup\{[\![\underline{s},\underline{6g-s}]\!]\colon 3\leqslant s\leqslant 6g-3\}$ *the augmented datum* $(g\cdot T,S,6g,3;[\![\underline{3,\ldots,3}]\!],[\![3,\ldots,3]\!],\pi_3)$ *is realizable.* This is easily shown by induction on g, using Fact 1 for the base and Fact 2 for the induction.

Fact 4. *If* $\mathcal{D}=(g\cdot T,S,d,3;\pi_1,\pi_2,[\![\underline{x},d-x]\!])$ *is realizable then*

$$\mathcal{D}'=(g\cdot T,S,d+2,3;[\![2]\!]*\pi_1,[\![2]\!]*\pi_2,[\![\underline{x+2},d-x]\!])$$

also is. To see this it is enough to take a dessin d'enfant Γ realizing $\mathcal{D}$ and to add one black vertex and one white vertex on an edge e having on both sides the complementary region of Γ corresponding to $\underline{x}$, as in Fig. 19.

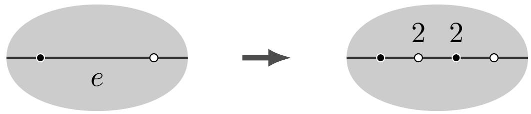

Figure 19: Fact 4 at the level of dessins d'enfant

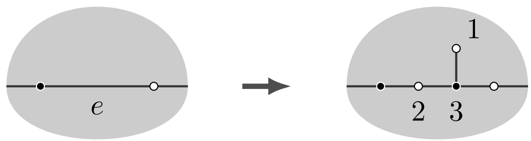

Figure 20: Fact 5 at the level of dessins d'enfant

FACT 5. *If* $\mathcal{D} = (g \cdot T, S, d, 3; \pi_1, \pi_2, [\![\underline{x}, d-x]\!])$ *is realizable then*

$$\mathcal{D}' = (g \cdot T, S, d+3, 3; [\![3]\!] * \pi_1, [\![1, 2]\!] * \pi_2, [\![\underline{x+3}, d-x]\!])$$

also is. In the usual framework, proceed as in Fig. 20.

FACT 6. *If* $\mathcal{D} = (g \cdot T, S, d, 3; \pi_1, \pi_2, [\![\underline{x}, d-x]\!])$ *is realizable then*

$$\mathcal{D}' = (g \cdot T, S, d+4, 3; [\![1, 3]\!] * \pi_1, [\![1, 3]\!] * \pi_2, [\![\underline{x+4}, d-x]\!])$$

also is. See Fig. 21.

CONCLUSION. It is now easy to see that each of the five candidate branch data of the statement can be obtained starting from the realizable augmented datum of Fact 3 by applying Facts 4, 5 and 6 zero or more times, and then forgetting about the augmentation. Namely:

1. No need to apply steps 4, 5 or 6;
2. Apply Fact 4;
3. Apply Fact 5;
4. Apply Fact 6;
5. Apply Facts 4 and 6. □

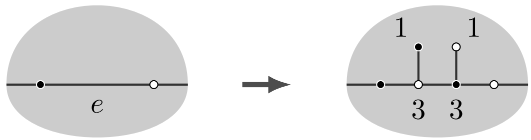

Figure 21: Fact 6 at the level of dessins d'enfant

Genus-1 covering surface We now prove the following:

Theorem 4.4. *The only exceptional candidate branch data of the form*

$$\mathcal{D} = (T, S, d, 3; \pi_1, \pi_2, [\![s, d-s]\!])$$

are items (7) to (11) in Theorem 3.1.

Proof. Setting $\ell_j = \ell(\pi_j)$ we see that the Riemann-Hurwitz condition (1) reads $\ell_1 + \ell_2 = d - 2$. First note that [5, Proposition 5.3] implies that no datum of the form

$$(T, S, d, n; \pi_1, \ldots, \pi_{n-1}, [\![1, d-1]\!])$$

is exceptional, so we can assume $1 < s < d-1$. Moreover a computer-aided analysis based on [33] shows that for $d \leqslant 16$ the only relevant exceptions are items (7) to (10) in Theorem 3.1 and item (11) for $5 \leqslant k \leqslant 8$, so we can assume $d \geqslant 17$ and we are left to show that only item (11) is exceptional.

The rest of the proof is split in the analysis of various cases.

Case 1: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1$, $\pi_2 = [\![\mathbf{2}]\!] * \rho_2$ with $x \geqslant 4$ and $\rho_2 \neq [\![2, \ldots, 2]\!]$. We apply to $\mathcal{D}$ the reduction move T_2 with $x_1 = 1$ and $x_2 = x - 3$ getting

$$\mathcal{D}' = (S, S, d-2, 3; [\![1, x-3]\!] * \rho_1, \rho_2, [\![s-1, d-s-1]\!])$$

(here and in the sequel when describing a case we already highlight the entries of π_1 and π_2 at which we will later apply a move). Since neither $[\![1, x-3]\!] * \rho_1$ nor ρ_2 can be $[\![2, \ldots, 2]\!]$, we see that $\mathcal{D}'$ is realizable by Theorem 4.1 unless it is item (6) with $d - 2 = kh$, $\rho_2 = [\![h, \ldots, h]\!]$ (so $h \geqslant 3$), $s - 1 = ph$ and either $\rho_1 = [\![1, \ldots, 1]\!]$ and $x - 3 = k + 1$ or $x - 3 = 1$ and $\rho_1 = [\![1, \ldots, 1, k+1]\!]$. Correspondingly, we have that $\mathcal{D}$ is one of the following:

$$(T, S, kh+2, 3; [\![1, \ldots, 1, \boldsymbol{k+4}]\!], [\![\mathbf{2}, h, \ldots, h]\!], [\![ph+1, (k-p)h+1]\!])$$
$$(T, S, kh+2, 3; [\![1, \ldots, 1, \mathbf{4}, k+1]\!], [\![2, \boldsymbol{h}, h, \ldots, h]\!], [\![ph+1, (k-p)h+1]\!]).$$

If we apply respectively the moves T_2 and T_4 at the highlighted entries of π_1 and π_2, we get

$$(S, S, kh, 3; [\![1, \ldots, 1, 2, k]\!], [\![h, \ldots, h]\!], [\![ph, (k-p)h]\!])$$
$$(S, S, kh, 3; [\![1, \ldots, 1, 2, k+1]\!], [\![2, h-2, h, \ldots, h]\!], [\![ph, (k-p)h]\!])$$

that are realizable by Theorem 4.1.

Case 2: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1, \pi_2 = [\![\boldsymbol{y}]\!] * \rho_2$ with $x, y \geqslant 4,\ \rho_1 \not\ni 2,\ \rho_2 \not\ni 2$. Then we apply T_4 getting

$$\mathcal{D}' = (S, S, d-2, 3; [\![x-2]\!] * \rho_1, [\![y-2]\!] * \rho_2, [\![s-1, d-s-1]\!])$$

which is realizable by Theorem 4.1 unless it is item (6) with (up to switch) $d-2 = kh,\ \rho_1 = [\![1, \ldots, 1]\!],\ x-2 = k+1,\ y-2 = h,\ \rho_2 = [\![h, \ldots, h]\!]$, whence $h \geqslant 3$, and $s-1 = ph$, so $\mathcal{D}$ is

$$(T, S, kh+2, 3; [\![1, \ldots, 1, \boldsymbol{k+3}]\!], [\![h+2, \boldsymbol{h}, h, \ldots, h]\!], [\![ph+1, (k-p)h+1]\!])$$

and applying T_4 we get the following datum which is realizable by Theorem 4.1:

$$(S, S, kh, 3; [\![1, \ldots, 1, k+1]\!], [\![h+2, h-2, h, \ldots, h]\!], [\![ph, (k-p)h]\!]).$$

Case 3: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1$ with $x \geqslant 4$, $\pi_2 = [\![\mathbf{3}]\!] * \rho_2$ with ρ_2 containing 1's and 3's only. Then we apply T_4 getting

$$(S, S, d-2, 3; [\![x-2]\!] * \rho_1, [\![1]\!] * \rho_2, [\![s-1, d-s-1]\!])$$

which by Theorem 4.1 is realizable unless it is item (6) with $d-2 = kh$, $x-2 = h$, $\rho_1 = [\![h, \ldots, h]\!]$, $\rho_2 = [\![1, \ldots, 1, k+1]\!]$, whence $k = 2$, and $s-1 = ph$. Then we have $p = 1$ and $d = 2h+2$, $h \geqslant 8$ and

$$\mathcal{D} = (T, S, 2h+2, 3; [\![h+2, h]\!], [\![3, \mathbf{3}, 1, 1, \ldots, 1]\!], [\![h+1, h+1]\!])$$

but applying T_1 we get the following datum which is realizable by Theorem 4.1:

$$(S, S, 2h+2, 3; [\![h+2, h]\!], [\![3, 1, \ldots, 1]\!], [\![h+1, h+1]\!]).$$

Case 4: $\max(\pi_1) = 3,\ \max(\pi_2) \leqslant 3,\ \pi_2 \neq [\![2, \ldots, 2]\!]$. We first note that if $\pi_1 \supseteq [\![1, 1]\!]$ we have $\pi_1 = [\![\mathbf{3}, 1, 1]\!] * \rho_1$ and we can apply T_1 getting

$$(S, S, d, 3; [\![1, 1, 1, 1, 1]\!] * \rho_1, \pi_2, [\![s, d-s]\!])$$

which can be exceptional only if it is item (6) with $k = 2$ and $h \leqslant 3$, hence $d \leqslant 6$, which we are excluding. So we assume $\pi_1 \not\supseteq [\![1, 1]\!]$ and we face the following subcases:

(i) $\pi_1 = [\![\mathbf{3}, \mathbf{3}]\!] * \rho_1,\ \pi_2 = [\![\mathbf{2}, \mathbf{2}]\!] * \rho_2$;

(ii) $\pi_2 \not\supseteq [\![2, 2]\!]$;

(iii) $\pi_1 \not\supseteq [\![3, 3]\!]$.

In subcase (i) for $s = 2$ Proposition 4.2 implies that $\mathcal{D}$ is realizable. Then we can assume $3 \leqslant s \leqslant d-3$ and apply T_3, getting

$$(S, S, d-4, 3; [\![1,1]\!] * \rho_1, \rho_2, [\![s-2, d-s-2]\!])$$

that can be exceptional only if it is item (6) with $k+1 \leqslant 3$, $h \leqslant 3$ and $kh = d-4$, whence $d \leqslant 10$, which we are excluding. In subcase (ii) we first note that π_2 contains 3's, otherwise it is $[\![2,1,\ldots,1]\!]$ and $\ell_2 = d-1$, but $\ell_1 + \ell_2 = d-2$. If $\pi_2 \supseteq [\![1,1]\!]$ we can switch the roles of π_1 and π_2 and use the first fact we noted to deduce that $\mathcal{D}$ is realizable. So we can assume $\pi_2 = [\![3,\ldots,3]\!] * \rho_2$ with $\rho_2 \subset [\![1,2]\!]$, and there are at least two 3's otherwise $d \leqslant 6$. This implies that if $\pi_1 \supseteq [\![2,2]\!]$ we are in case (i) with roles switched, and $\mathcal{D}$ is realizable. Otherwise we also have $\pi_1 = [\![3,\ldots,3]\!] * \rho_1$ with $\rho_1 \subset [\![1,2]\!]$, hence for $j = 1,2$ we have $\ell_j \leqslant (d-(1+2))/3 + 2 = d/3 + 1$, but then

$$d - 2 = \ell_1 + \ell_2 \leqslant \frac{2}{3}d + 2 \;\Rightarrow\; d \leqslant 12.$$

Finally in subcase (iii) we claim that $\pi_2 \supseteq [\![3,3]\!]$, otherwise for $j = 1,2$ we have $\ell_j \geqslant (d-3)/2 + 1$ whence the contradiction

$$d - 2 = \ell_1 + \ell_2 \geqslant (d-3) + 2 = d - 1.$$

Then, switching roles, we are in subcase (i) or (ii), so we conclude that $\mathcal{D}$ is realizable.

CASE 5: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1$ with $x = \max(\pi_1) \geqslant 4$, $\pi_2 = [\![\mathbf{2}, 2, \ldots, 2]\!]$. We apply move T_2 with $x_1 = 1$ and $x_2 = x - 3$, getting

$$(S, S, d-2, 3; [\![1, x-3]\!] * \rho_1, [\![2,\ldots,2]\!], [\![s-1, d-s-1]\!]).$$

This can be one of the items (1)-(6) in many different ways, namely:

(3) with $d - 2 = 2k$, $s - 1 = k$ and
   - (a) $x - 3 = 2$, $\rho_1 = [\![2,\ldots,2,3]\!]$
   - (b) $x - 3 = 3$, $\rho_1 = [\![2,\ldots,2]\!]$

(4) with $d - 2 = 4k + 2$, whence $k \geqslant 4$, $s - 1 = 2k + 1$ and
   - (a) $x - 3 = k + 1$, $\rho_1 = [\![1,\ldots,1,k+2]\!]$
   - (b) $x - 3 = k + 2$, $\rho_1 = [\![1,\ldots,1,k+1]\!]$

(5) with $d - 2 = 4k$, whence $k \geqslant 4$, $s - 1 = 2k - 1$ and $x - 3 = k + 1$, $\rho_1 = [\![1,\ldots,1,k+1]\!]$

(6) with $h = 2$, $d - 2 = 2k$, whence $k \geqslant 8$, $s - 1 = 2p$ and $x - 3 = k + 1$, $\rho_1 = [\![1,\ldots,1]\!]$.

Correspondingly we see that $\mathcal{D}$ is

(3-a) $(T, S, 2k+2, 3; [\![2, \ldots, 2, 3, 5]\!], [\![2, \ldots, 2]\!], [\![k+1, k+1]\!])$

(3-b) $(T, S, 2k+2, 3; [\![2, \ldots, 2, \mathbf{6}]\!], [\![2, \ldots, 2]\!], [\![k+1, k+1]\!])$

(4-a) $(T, S, 4k+4, 3; [\![1, \ldots, 1, k+2, \boldsymbol{k+4}]\!], [\![2, \ldots, 2]\!], [\![2k+2, 2k+2]\!])$

(4-b) $(T, S, 4k+4, 3; [\![1, \ldots, 1, k+1, \boldsymbol{k+5}]\!], [\![2, \ldots, 2]\!], [\![2k+2, 2k+2]\!])$

(5) $(T, S, 4k+2, 3; [\![1, \ldots, 1, k+1, \boldsymbol{k+4}]\!], [\![2, \ldots, 2]\!], [\![2k, 2k+2]\!])$

(6) $(T, S, 2k+2, 3; [\![1, \ldots, 1, \boldsymbol{k+4}]\!], [\![2, \ldots, 2]\!], [\![2p+1, 2(k-p)+1]\!])$.

Now (3-a) is precisely the exceptional item (11) of the statement. In all the other cases we perform a reduction move T_2 at the highlighted entry of π_1, always with $x_1 = 2$, getting a realizable candidate branch datum, namely one that cannot be one of items (1) to (6):

(3-b) $(S, S, 2k, 3; [\![2, \ldots, 2]\!], [\![2, \ldots, 2]\!], [\![k, k]\!])$

(4-a) $(S, S, 4k+2, 3; [\![1, \ldots, 1, 2, k, k+2]\!], [\![2, \ldots, 2]\!], [\![2k+1, 2k+1]\!])$

(4-b) $(S, S, 4k+2, 3; [\![1, \ldots, 1, 2, k+1, k+1]\!], [\![2, \ldots, 2]\!], [\![2k+1, 2k+1]\!])$

(5) $(S, S, 4k, 3; [\![1, \ldots, 1, 2, k, k+1]\!], [\![2, \ldots, 2]\!], [\![2k-1, 2k+1]\!])$

(6) $(S, S, 2k, 3; [\![1, \ldots, 1, 2, k]\!], [\![2, \ldots, 2]\!], [\![2p, 2(k-p)]\!])$.

Case 6: $\max(\pi_1) = 3$, $\pi_2 = [\![2, \ldots, 2]\!]$. If $d = 2k$ we have $\ell_2 = k$, whence $\ell_1 = k-2$, which easily implies that $\pi_1 = [\![3, 3, \mathbf{3}, \mathbf{3}]\!] * \rho_1$. If $s = 2$ or $s = d-2$ Proposition 4.2 implies that $\mathcal{D}$ is realizable. Otherwise we apply a move T_3 getting

$$(S, S, 2k-4, 3; [\![3, 3, 1, 1]\!] * \rho_1, [\![2, \ldots, 2]\!], [\![s-2, d-s-2]\!])$$

which is realizable unless it is item (5), but we are assuming $d > 8$.

Conclusion: Knowing that $\ell_1 + \ell_2 = d-2$ one easily sees that the above cases cover all possibilities up to switching π_1 and π_2. □

Large genus covering surface We now prove the following:

Theorem 4.5. *For $g \geqslant 2$ there is no exceptional candidate branch datum of the form*

$$\mathcal{D} = (g \cdot T, S, d, 3; \pi_1, \pi_2, [\![s, d-s]\!]).$$

Proof. By induction on $g \geqslant 1$ we prove that the only exceptional $\mathcal{D}$ as in the statement are items (7) to (11) in Theorem 3.1. The base step $g = 1$ is Theorem 4.4. Now we assume $g \geqslant 2$ and we do the inductive step. Again [5, Proposition 5.3] implies that we can assume $2 \leqslant s \leqslant d-2$. Moreover again a computer-aided analysis based on [33] shows that for $d \leqslant 20$ we have no exceptions, so we assume $d \geqslant 21$. We now analyse various cases showing that there always exists a reduction move $T_j : \mathcal{D} \rightsquigarrow \mathcal{D}'$ such that $\mathcal{D}'$ is not item (11) in Theorem 3.1, which is enough. Note that if $\ell_j = \ell(\pi_j)$ we have $\ell_1 + \ell_2 = d - 2g$.

Case 1: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1$, $\pi_2 = [\![\mathbf{2}]\!] * \rho_2$ with $x \geq 4$. We apply T_2 with $x_1 = 1$ and $x_2 = x - 3$, getting the desired

$$\mathcal{D}' = ((g-1) \cdot T, S, d-2, 3; [\![1, x-3]\!] * \rho_1, \rho_2, [\![s-1, d-s-1]\!]).$$

Case 2: $\pi_1 = [\![\boldsymbol{x}]\!] * \rho_1, \pi_2 = [\![\boldsymbol{y}]\!] * \rho_2$, with $x \geqslant 4, y \geqslant 3$, and $\pi_2 \not\ni 2$. Then we can apply T_4 getting the desired

$$\mathcal{D}' = ((g-1) \cdot T, S, d-2, 3; [\![x-2]\!] * \rho_1, [\![y-2]\!] * \rho_2, [\![s-1, d-s-1]\!]).$$

Case 3: $\max(\pi_1) = 3$ and $\max(\pi_2) \leqslant 3$. Here we further distinguish some situations:

(a) $\pi_1 \supseteq [\![1, 1]\!]$, hence $\pi_1 = [\![\mathbf{3}, 1, 1]\!] * \rho_1$. Then we apply T_1 getting the desired

$$\mathcal{D}' = ((g-1) \cdot T, S, d, 3; [\![1, 1, 1, 1, 1]\!] * \rho_1, \pi_2, [\![s, d-s]\!])$$

(b) $\pi_1 = [\![\mathbf{3}, \mathbf{3}]\!] * \rho_1$, $\pi_2 = [\![2, 2]\!] * \rho_2$; for $s = 2$ or $d - 2$ we conclude that $\mathcal{D}$ is realizable by Proposition 4.2, otherwise we apply T_3 getting the desired

$$\mathcal{D}' = ((g-1) \cdot T, S, d-4, 3; [\![1, 1]\!] * \rho_1, \rho_2, [\![s-2, d-s-2]\!])$$

(c) $\pi_1 \not\supseteq [\![2, 2]\!]$. If $\pi_1 \supseteq [\![1, 1]\!]$ we are in case (3-a), so we can assume $\pi_1 \not\supseteq [\![1, 1]\!]$. Then of course $\pi_1 \supseteq [\![3, 3]\!]$. Now if $\pi_2 \supseteq [\![2, 2]\!]$ we are in case (3-b), so we can assume $\pi_2 \not\supseteq [\![2, 2]\!]$. If $\pi_2 \not\supseteq [\![3, 3]\!]$ we have $\ell_2 \geqslant d - 3$, but $\ell_1 \geqslant d/3$, which combined with $\ell_1 + \ell_2 = d - 2g$ gives $d + 6g \leqslant 9$, which we are excluding. So we can assume $\pi_2 \supseteq [\![3, 3]\!]$, and again by case (3-b), switching roles, we can also assume that $\pi_1 \not\supseteq [\![2, 2]\!]$. We then have that $\pi_j = [\![3, \ldots, 3]\!] * \rho_j$ with $\rho_j \subseteq [\![1, 2]\!]$ for $j = 1, 2$, whence $\mathcal{D}$ is realizable by Proposition 4.3

(d) $\pi_1 \not\supseteq [\![3, 3]\!]$. This implies that $\ell_1 \geqslant (d-3)/2 + 1$, namely $\ell_1 \geqslant (d-1)/2$. If also $\pi_2 \not\supseteq [\![3, 3]\!]$ then $\ell_2 \geqslant (d-1)/2$ as well, which contradicts $\ell_1 + \ell_2 = d - 2g$. So $\pi_2 \supseteq [\![3, 3]\!]$, but we can assume $\pi_1 \supseteq [\![2, 2]\!]$ by case (3-c), so we are in case (3-b) with roles switched, and again we conclude that $\mathcal{D}$ is realizable

(e) If none of the above holds, in particular $\pi_1 \supseteq [\![2, 2]\!]$ and $\pi_2 \not\supseteq [\![2, 2]\!]$. Now if $\pi_2 \supseteq [\![3, 3]\!]$ we are in case (3-b) with roles switched, so we can assume $\pi_2 \not\supseteq [\![3, 3]\!]$, therefore $\ell_2 \geqslant d - 3$, but $\ell_1 \geqslant d/3$, which as above is excluded.

Conclusion: We cannot have $\max(\pi_1) = \max(\pi_2) = 2$ otherwise $\ell_1, \ell_2 \geqslant d/2$, but $\ell_1 + \ell_2 = d - 2g$. So either, up to switching, $\max(\pi_1) \geqslant 4$ or $\max(\pi_1) = 3$ and $\max(\pi_2) \leqslant 3$. The latter situation is Case 3. In the former either $\pi_2 \ni 2$, whence Case 1, or $\pi_2 \not\ni 2$, but π_2 is non-trivial, so $\max(\pi_2) \geqslant 3$, and we are in Case 2. □

5 Realizability for more than three branching points

We begin by citing [5, Complement 5.6]:

Proposition 5.1. *The only exceptional branch datum*

$$\mathcal{D} = (g \cdot T, S, 4, n; \pi_1, \ldots, \pi_n)$$

is item (13) in Theorem 3.1.

The next result eventually completes the proof of Theorem 3.1:

Theorem 5.2. *The only exceptional data*

$$\mathcal{D} = (g \cdot T, S, d, n; \pi_1, \ldots, \pi_{n-1}, [\![s, d-s]\!])$$

with $n \geqslant 4$ are items (12) and (13) in Theorem 3.1.

Proof. Within this proof we use the notation of Section 2. We proceed by induction on n. The base step $n = 4$ requires some work. First of all a computer-aided analysis based on [33] implies that for $d \leqslant 16$ the only exceptional $\mathcal{D}$ as in the statement are item (12) and item (13) with $n = 4$, so we assume $d \geqslant 17$. Set $v_j = v(\pi_j)$.

Case 1: $v_1 + v_2 < d$. We can then apply the reduction move A_1 at π_1 and π_2, taking θ_1 and θ_2 as given by Proposition 2.2, getting

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (g \cdot T, S, d, 3; \pi, \pi_3, [\![s, d-s]\!])$$

which is realizable unless it is item (11), namely

$$\mathcal{D}' = (T, S, 2k, 3; [\![2, \ldots, 2]\!], [\![2, \ldots, 2, 3, 5]\!], [\![k, k]\!]),$$

so $g = 1$, $d = 2k$ and $s = k$, with $k > 8$. Moreover either

(I) $\pi = [\![2, \ldots, 2]\!]$, $\pi_3 = [\![2, \ldots, 2, 3, 5]\!]$, so $v_1 + v_2 = k$, $v_3 = k + 2$, or

(II) $\pi = [\![2, \ldots, 2, 3, 5]\!]$, $\pi_3 = [\![2, \ldots, 2]\!]$, so $v_1 + v_2 = k + 2$, $v_3 = k$.

In case (I) we can suppose $v_1 \leqslant k/2$, hence

$$k + 2 < 1 + k + 2 \leqslant v_1 + v_3 \leqslant 3k/2 + 2 < 2k = d$$

so we can apply Proposition 2.2 and the reduction move A_1 to $\mathcal{D}$ at π_1 and π_3 getting an analogous $\mathcal{D}'$ with $v(\pi) > k+2$, which cannot be item (11), so it is realizable.

In case (II) instead, noting that $v_1 + v_2 = k+2$ we see that the following cases cover all the possibilities up to switching indices:

(a) $v_1 < k$ and $v_1 \neq 2$

(b) $v_1 = 2$ and $\pi_2 \neq [\![2, \ldots, 2]\!]$

(c) $v_1 = 2$ and $\pi_2 = [\![2, \ldots, 2]\!]$.

In case (a), since $v_3 = k$ we have $v_1 + v_3 < 2k = d$, so we can apply Proposition 2.2 and the reduction move A_1 to $\mathcal{D}$ at π_1 and π_3, getting $\mathcal{D}' = (g \cdot T, S, 2k, 3; \pi, \pi_2, [\![k, k]\!])$ with $v(\pi) = v_1 + k$, that cannot be k or $k+2$, so $\mathcal{D}'$ is realizable. In case (b) we have $v_1 + v_3 = 2 + k < 2k = d$, so again we can apply Proposition 2.2 and the reduction move A_1 to $\mathcal{D}$ at π_1 and π_3, getting $\mathcal{D}' = (g \cdot T, S, 2k, 3; \pi, \pi_2, [\![k, k]\!])$ with $v_2 = k$ but $\pi_2 \neq [\![2, \ldots, 2]\!]$, so $\mathcal{D}'$ is realizable. In case (c) note that $\pi_1 = [\![1, \ldots, 1, 2, 2]\!]$ or $\pi_1 = [\![1, \ldots, 1, 3]\!]$. Here we apply Proposition 2.5 in its full strength, namely choosing θ_1 and θ_2. In both cases we take $\theta_2 = (1,2)(3,4)\cdots(2k-1, 2k)$, while $\theta_1 = (1,3)(2,5)$ for $\pi_1 = [\![1, \ldots, 1, 2, 2]\!]$ and $\theta_1 = (1,3,5)$ for $\pi_1 = [\![1, \ldots, 1, 3]\!]$. Then $\theta_1 \cdot \theta_2$ is respectively

$$(1,5,6,2,3,4)(7,8)\cdots(2k-1,2k) \qquad (1,2,3,4,5,6)(7,8)\cdots(2k-1,2k)$$

whence

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (T, S, 2k, 3; [\![6, 2, \ldots, 2]\!], [\![2, \ldots, 2]\!], [\![k, k]\!])$$

which is realizable.

Case 2: $v_i + v_j \geqslant d$ for all $1 \leqslant i < j \leqslant 3$. Without loss of generality we can assume $v_3 \geqslant d/2$. Noting that

$$v_1 + v_2 \geqslant d \qquad v_3 + v([\![s, d-s]\!]) \geqslant 1 + d - 2 = d - 1$$

we can apply move A_2 to $\mathcal{D}$ at π_1 and π_2, getting

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (g' \cdot T, S, d, 3; \pi, \pi_3, [\![s, d-s]\!])$$

with $v(\pi) \geqslant d-2$. We can now compute

$$\begin{aligned} g' &= \frac{1}{2}(v(\pi) + v_3 + v([\![s, d-s]\!])) - d + 1 \\ &\geqslant \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geqslant 2, \end{aligned}$$

so $\mathcal{D}'$ is realizable by Theorem 4.5 and the base step $n = 4$ is complete.

For the induction step, we assume $n \geqslant 5$ and we take $\mathcal{D}$ as in the statement. If $d = 4$ the conclusion follows from Proposition 5.1, so we assume $d \neq 4$. Now if up to permutation we have $v_1 + v_2 \leqslant d - 1$ we can apply a reduction move A_1 (also making use of Proposition 2.2) to $\mathcal{D}$ at π_1 and π_2, getting $\mathcal{D}'$ with $n - 1$ non-trivial partitions. Otherwise we have $v_1 + v_2 \geqslant d - 1$ and $v_3 + v_4 \geqslant d - 1$, so we can apply a reduction move A_2 to $\mathcal{D}$ at π_1 and π_2, getting again $\mathcal{D}'$ with $n - 1$ non-trivial partitions. By induction, such a $\mathcal{D}'$ can only be exceptional if it is item (12) in Theorem 3.1, so $n - 1 = 4$ and $d = 8$. But for $n = 5$ and $d = 8$ a computer-aided search based on [33] implies that $\mathcal{D}$ is not exceptional. □

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